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Another route to the three-dimensional development of Tollmien–Schlichting waves with finite amplitude

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The Tollmien–Schlichting waves appearing as a result of instability of laminar flows develop a three-dimensional configuration as the amplitude becomes large enough. A new explanation of this experimentally observed phenomenon is attempted on the basis of a resonance theory. It is shown that the existence of two-dimensional waves with finite amplitude can induce three-dimensional distortion with spanwise periodicity of the mean-flow field. Under a certain condition for resonance, the distortion grows, in proportion to the product of time and an exponential function of time, up to quite a large magnitude, and consequently interacts with the Tollmien–Schlichting waves to yield new three-dimensional travelling waves with the same streamwise wavenumber as the two-dimensional waves, and with the same spanwise wavenumber as the mean flow. The resulting flow field is of the peak–valley-splitting type, as observed often in experiments. The growth rate of the three-dimensional part in the mean flow depends significantly upon values of the spanwise wavenumber, suggesting that there is a preferred range of spanwise periodicity in the three-dimensional development of unstable laminar flows.

1. Introduction

The transition from laminar flow to turbulence in two-dimensional flows, such as the Blasius boundary layer and plane Poiseuille flow, begins with the evolution of two-dimensional travelling waves, namely Tollmien–Schlichting waves, when naturally existing disturbances are sufficiently small, or when weak two-dimensional excitation is introduced through the vibrating-ribbon technique. These waves grow during propagation in the downstream direction and, after achieving a certain magnitude, develop a three-dimensional configuration with periodicity in the spanwise direction. It is widely recognized that this spanwise inhomogeneity is indispensable for the subsequent process of transition to turbulence. Thus the mechanics leading to three-dimensionalization of Tollmien–Schlichting waves has been one of the most important subjects to arise in stability and transition research. An excellent review of this subject is given by Herbert & Morkovin (1980) (see also Herbert 1984*b*).

As seen in the experiments by Klebanoff, Tidstrom & Sargent (1962), the wave disturbances exhibit clear three-dimensionality after their growth rate separates from the prediction of linear stability theory, suggesting that the distortion is mainly due to nonlinear effects of finite-amplitude disturbances (Tani 1969). The first attempt to describe the spanwise-periodic configuration on the basis of nonlinear theory was made by Benney & Lin (1960), but their theory is concerned only with the mean-flow field induced by interaction between two- and three-dimensional waves existing beforehand. For the purpose of seeking an origin of the three-dimensionality, the work of Stuart (1962) seems to be more important. He extended the weakly nonlinear stability theory of Stuart (1960) and Watson (1960) to describe interactions of two- and three-dimensional travelling waves, and derived the amplitude equations governing nonlinear growth of the two waves. If the amplitude of three-dimensional waves is taken sufficiently small, his theory is usable for the present purpose. Some important aspects of such disturbances were revealed by Itoh (1980) through numerical evaluation of various coefficients of the amplitude equations.

In the above formulation, the two- and three-dimensional waves are assumed to have the same wavenumber in the streamwise direction. Since linear stability theory gives different values of phase velocity for the two waves, no resonance between them can occur, at least in the lowest-order approximation, in contradiction to experimental observations. In such circumstances, Craik (1971) directed his attention to three-dimensional waves with a streamwise wavenumber just half of the two-dimensional one (see also Raetz 1959), and found that the waves with a particular spanwise wavenumber could have the same phase velocity as the two-dimensional one, resulting in a kind of resonance accompanied by strong growth of all the waves. The disturbances predicted by this theory comprise the so-called resonant triads, which were quite recently detected in experiments by Saric & Thomas (1984) and Kachanov & Levchenko (1984).

Another theoretical study concerning this subject has been made recently by Herbert (1984*a*), who considered the time-periodic flow consisting of a steady laminar flow plus two-dimensional waves with finite and equilibrium amplitude, and revealed its instability to infinitesimal three-dimensional disturbances with spanwise wavenumber over a rather wide range. His analysis is novel in that the disturbances considered include a number of components in the streamwise Fourier expansion.

The present paper aims at proposing a possible solution to the problem of three-dimensionalization. First, a general formulation of the problem is given, followed by a more detailed discussion of the three theories introduced above to clarify our present state of understanding about the problem. In §4, a mathematical identification is made of various kinds of resonance in physically different situations, in relation to the motivation and basic idea of the present work. The last two sections are devoted to the description of a new theory of resonance in the mean-flow field and a discussion of the theoretical and numerical results.

2. The fundamental equations

For mathematical simplicity, we consider a flow of incompressible fluid between parallel planes, and let x denote the coordinate in the flow direction, y the spanwise coordinate and z the coordinate normal to the planes at $z = \pm 1$, with u, v, w as corresponding velocity components ($\boldsymbol{v}$ in vector form), and t the time. Here all quantities have been made non-dimensional with the half-width h of the channel and the maximum velocity U_0 of the steady laminar flow (plane Poiseuille flow), so that the Reynolds number is defined as $R = U_0 h/\nu$, ν being the kinematic viscosity.

The problem is to investigate stability of the basic flow consisting of the steady laminar flow $\boldsymbol{V} = (U, 0, 0)$ plus the two-dimensional travelling waves $\hat{\boldsymbol{V}} = (\hat{U}, 0, \hat{W})$ to some kinds of three-dimensional disturbances with velocity $\hat{\boldsymbol{v}} = (\hat{u}, \hat{v}, \hat{w})$. The basic flow must satisfy the Navier–Stokes equations, and so the steady laminar part is given by $U(z) = 1-z^2$, while the wave part is governed by the nonlinear equation

$$\left\{\left(\frac{1}{R}\nabla^2-\frac{\partial}{\partial t}-U\frac{\partial}{\partial x}\right)\nabla^2+U''\frac{\partial}{\partial x}\right\}\Psi=\left(\frac{\partial\Psi}{\partial z}\frac{\partial}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial z}\right)\nabla^2\Psi, \tag{2.1}$$

where $\nabla^2 \equiv \partial^2/\partial x^2+\partial^2/\partial z^2$, $' \equiv \partial/\partial z$, and the stream function $\Psi(x,z,t)$ has been introduced with the definition

$$\hat{U}=\frac{\partial\Psi}{\partial z},\quad \hat{W}=-\frac{\partial\Psi}{\partial x}. \tag{2.2}$$

The above equation admits a wave solution of the form

$$\Psi=\sum_{k=-\infty}^{\infty}\Psi_k(z,t)\,\mathrm{e}^{\mathrm{i}k\alpha(x-c_r t)},\quad \Psi_{-k}=\tilde{\Psi}_k, \tag{2.3}$$

where the real constants α and c_r denote the wavenumber and phase velocity respectively, while the tilde indicates the complex conjugate. In the present formulation, the temporally dependent approach is used for mathematical convenience, and so the Fourier coefficients in the above are assumed to be functions of z and t, although real waves observed in experiments seem to vary in the streamwise direction. Substituting (2.3) into (2.1) and separating out every Fourier component, we obtain an infinite sequence of equations for Ψ_k, with $k = 0, \pm 1, \pm 2, \ldots$, as

$$\begin{aligned}\left[\left\{\frac{1}{R}(\mathrm{D}^2-k^2\alpha^2)+\mathrm{i}k\alpha c_r-\frac{\partial}{\partial t}-\mathrm{i}k\alpha U\right\}(\mathrm{D}^2-k^2\alpha^2)+\mathrm{i}k\alpha U''\right]\Psi_k&\\ =\sum_{l=-\infty}^{\infty}\{\mathrm{i}l\alpha\mathrm{D}\Psi_{k-l}-\mathrm{i}(k-l)\alpha\Psi_{k-l}\mathrm{D}\}(\mathrm{D}^2-l^2\alpha^2)\Psi_l,&\end{aligned} \tag{2.4}$$

where $\mathrm{D} \equiv \partial/\partial z$. It is not easy to solve these partial differential equations simultaneously, and therefore so far some approximate methods have been used. In general, the Fourier series is truncated at an appropriate term $k = \pm K$, and then the truncated system can be solved numerically if an appropriate initial condition is specified. In a particular case where the waves may be assumed to be in equilibrium, however, no initial conditions need to be specified and computations become much simpler because the equations reduce to ordinary differential ones. On the other hand, if the waves are sufficiently small, it is possible to apply the weakly nonlinear approach, in which the Fourier coefficients are expanded into power series of a suitably defined amplitude. This last method has the advantage that the equations can be solved successively up to an arbitrary level of approximation, but has the great disadvantage that the radius of convergence of the series solution is very small (see Herbert 1980). Any of the three methods above may be used according to the purpose of the study, but great care should be taken that the approximation introduced at this stage does not induce any essential error in the final results of the calculations.

We now proceed to derive the equations governing a small three-dimensional disturbance superimposed on the two-dimensional unsteady flow. We substitute $\boldsymbol{v} = \boldsymbol{V}+\hat{\boldsymbol{V}}+\hat{\boldsymbol{v}}$ into the Navier–Stokes and continuity equations, subtract the basic-flow parts, and linearize the resultant equations with respect to $\hat{\boldsymbol{v}}$. The disturbance is assumed to be periodic in both the streamwise and spanwise directions, so as to be expanded into the double Fourier series. Since the equations have been linearized, we may take only a single component of the y-Fourier series, although components in the x-Fourier series should be retained in the way similar to the case of two-dimensional waves. Thus we put

$$\left.\begin{aligned} \hat{u} &= \cos\beta y \sum_{k=-\infty}^{\infty} u_k(z,t)\, \mathrm{e}^{\mathrm{i}k\alpha(x-c_\mathrm{r}t)}, \quad u_{-k} = \tilde{u}_k, \\ \hat{v} &= \sin\beta y \sum_{k=-\infty}^{\infty} v_k(z,t)\, \mathrm{e}^{\mathrm{i}k\alpha(x-c_\mathrm{r}t)}, \quad v_{-k} = \tilde{v}_k, \\ \hat{w} &= \cos\beta y \sum_{k=-\infty}^{\infty} w_k(z,t)\, \mathrm{e}^{\mathrm{i}k\alpha(x-c_\mathrm{r}t)}, \quad w_{-k} = \tilde{w}_k, \end{aligned}\right\} \tag{2.5}$$

where the real constant β denotes the wavenumber in the spanwise direction. These are substituted into the linear disturbance equations with pressure terms eliminated, and each Fourier component is separated out to yield

$$\begin{aligned} &\left[\left\{\frac{1}{R}(\mathrm{D}^2-k^2\alpha^2-\beta^2)+\mathrm{i}k\alpha c_\mathrm{r}-\frac{\partial}{\partial t}-\mathrm{i}k\alpha U\right\}(\mathrm{D}^2-k^2\alpha^2-\beta^2)+\mathrm{i}k\alpha U''\right]w_k \\ &\quad = -(k^2\alpha^2+\beta^2)\sum_{l=-\infty}^{\infty}[\{\mathrm{i}l\alpha \mathrm{D}\Psi_{k-l}-\mathrm{i}(k-l)\,\alpha\Psi_{k-l}\,\mathrm{D}\}\,w_l-\mathrm{i}l\alpha(\mathrm{i}l\alpha u_{k-l}+w_{k-l}\,\mathrm{D})\,\Psi_l] \\ &\qquad -\mathrm{D}\sum_{l=-\infty}^{\infty}[\{\mathrm{i}l\alpha \mathrm{D}\Psi_{k-l}-\mathrm{i}(k-l)\,\alpha\Psi_{k-l}\,\mathrm{D}\}\{\mathrm{i}(k-l)\,\alpha u_l-\mathrm{D}w_l\} \\ &\qquad\qquad +\mathrm{i}k\alpha(\mathrm{i}l\alpha u_{k-l}+w_{k-l}\,\mathrm{D})\,\mathrm{D}\Psi_l], \end{aligned} \tag{2.6}$$

$$\begin{aligned} &\left\{\frac{1}{R}(\mathrm{D}^2-k^2\alpha^2-\beta^2)+\mathrm{i}k\alpha c_\mathrm{r}-\frac{\partial}{\partial t}-\mathrm{i}k\alpha U\right\}(k^2\alpha^2+\beta^2)\,u_k \\ &\quad -\left[\mathrm{i}k\alpha\left\{\frac{1}{R}(\mathrm{D}^2-k^2\alpha^2-\beta^2)+\mathrm{i}k\alpha c_\mathrm{r}-\frac{\partial}{\partial t}-\mathrm{i}k\alpha U\right\}\mathrm{D}+\beta^2U'\right]w_k \\ &\quad = \sum_{l=-\infty}^{\infty}[\{\mathrm{i}l\alpha \mathrm{D}\Psi_{k-l}-\mathrm{i}(k-l)\,\alpha\Psi_{k-l}\,\mathrm{D}\}\{(kl\alpha^2+\beta^2)\,u_l-\mathrm{i}k\alpha \mathrm{D}w_l\} \\ &\qquad\qquad +\beta^2(\mathrm{i}l\alpha u_{k-l}+w_{k-l}\,\mathrm{D})\,\mathrm{D}\Psi_l], \end{aligned} \tag{2.7}$$

$$\mathrm{i}k\alpha u_k+\beta v_k+\mathrm{D}w_k = 0, \tag{2.8}$$

where $k = 0, 1, 2, \ldots$. Ultimately, the problem to be attacked is to solve these equations subject to the boundary conditions

$$u_k = v_k = w_k = \mathrm{D}w_k = 0 \quad \text{at } z = \pm 1. \tag{2.9}$$

3. Existing theories

The three theories briefly mentioned in §1 may be thought of as presenting different kinds of approximate solutions to the problem posed in §2. Before developing a new theory, it will be useful background knowledge to discuss more details of these theories in line with our formulation.

3.1. *The weakly nonlinear theory of Stuart and Itoh*

In the original theory of Stuart (1962) (see also Itoh 1980), two- and three-dimensional waves are assumed to be of the same small order, but we consider here only the simpler case, of the infinitesimally small three-dimensional wave. Let us assume the fundamental component Ψ_1 in (2.3) to be dominant over the others, and denote its complex amplitude by $A(t)$. Then the weakly nonlinear solution to (2.4) is written in the form

$$\Psi_k = A^k \sum_{n=0}^{\infty} |A|^{2n}\, \Phi_k^{(n)}(z), \tag{3.1}$$

$$\frac{\mathrm{d}A}{\mathrm{d}T} = -\mathrm{i}A\left(\mathrm{i}\alpha c_\mathrm{i} + \sum_{n=1}^{\infty} A_n |A|^{2n}\right), \tag{3.2}$$

where $k = 0, 1, 2, \ldots$, $\Phi_0^{(0)} \equiv 0$, and $\Phi_1^{(n)}$ for $n = 0, 1, 2, \ldots$ are normalized as

$$\Phi_1^{(0)}(0) = 1, \quad \Phi_1^{(n)}(0) = 0 \quad (n \geqslant 1), \tag{3.3}$$

indicating that the amplitude A represents the value of Ψ_1 at $z = 0$. Coefficients of the above series in powers of $|A|^2$ are obtainable successively according to the well-known procedure of weakly nonlinear theory, which begins with the Orr–Sommerfeld eigenvalue problem to determine the complex constant $c \equiv c_\mathrm{r} + \mathrm{i}c_\mathrm{i}$ and the complex function $\Phi_1^{(0)}(z)$, followed by the solution of the inhomogeneous equations for $\Phi_0^{(1)}(z)$ and $\Phi_2^{(0)}(z)$. The series solution thus obtained converges for sufficiently small values of $|A|$.†

A similar expansion in terms of the amplitude A is applied to the stability problem concerned here. We retain only the fundamental components in the Fourier series (2.5), neglecting the others as smaller-order terms in $|A|$, and put

$$\boldsymbol{v}_1 = B\left\{\boldsymbol{v}_1^{(0)}(z) + |A|^2 \boldsymbol{v}_1^{(1)}(z) + \frac{A^2\tilde{B}}{B}\hat{\boldsymbol{v}}_1^{(1)}(z) + O(|A|^4)\right\}, \tag{3.4}$$

$$\frac{\mathrm{d}B}{\mathrm{d}t} = -\mathrm{i}B\left\{\alpha(\hat{c} - c_\mathrm{r}) + \lambda_1 |A|^2 + \hat{\lambda}_1 \frac{A^2\tilde{B}}{B} + O(|A|^4)\right\}, \tag{3.5}$$

where $B(t)$ denotes the complex amplitude of the three-dimensional wave, and the associated normalization is made on w_1 in a form similar to (3.3). Substitution of these into (2.6)–(2.9) and separation of various powers of amplitudes leads to a sequence of ordinary differential equations and boundary conditions. From the lowest-order terms, we have the homogeneous equation

$$\left\{\frac{1}{R}(\mathrm{D}^2 - \alpha^2 - \beta^2)^2 - \mathrm{i}\alpha(U - \hat{c})\,(\mathrm{D}^2 - \alpha^2 - \beta^2) + \mathrm{i}\alpha U''\right\} w_1^{(0)} = 0, \tag{3.6}$$

which, together with homogeneous boundary conditions, determines the complex eigenvalue $\hat{c}$ and the corresponding eigenfunction $w_1^{(0)}(z)$ as functions of the Reynolds number R and the wavenumbers α and β. The associated velocity components in the streamwise and spanwise directions are obtained from solution of the inhomogeneous equations with the now-known forcing terms.

On the other hand, the next-order terms in the amplitude expansion give rise to the following inhomogeneous equations of a different type:

$$\left\{\frac{1}{R}(\mathrm{D}^2 - \alpha^2 - \beta^2)^2 - \mathrm{i}\alpha(U - \hat{c} - 2\mathrm{i}c_\mathrm{i})\,(\mathrm{D}^2 - \alpha^2 - \beta^2) + \mathrm{i}\alpha U''\right\} w_1^{(1)}$$
$$= \{\mathrm{i}\alpha D\Phi_0^{(1)}(\mathrm{D}^2 - \alpha^2 - \beta^2) - \mathrm{i}\alpha \mathrm{D}^3\Phi_0^{(1)} - \mathrm{i}\lambda_1(\mathrm{D}^2 - \alpha^2 - \beta^2)\}\, w_1^{(0)}, \tag{3.7}$$

† The formulation given here works successfully if the growth rate c_i is not negative, but may possibly break down in the case of c_i negative because the equations for $\Phi_0^{(n)}$ $(n \geqslant 1)$ involve a kind of singularity, as clearly explained by Herbert (1983) (see also Itoh 1974*b*). However, particular solutions of finite equilibrium amplitude can be obtained by using the false-problem method, where the condition $\mathrm{Re}\,[\mathrm{d}A/\mathrm{d}t] = 0$ is imposed from the beginning (Reynolds & Potter 1967; Itoh 1977).

$$\left\{\frac{1}{R}(\mathrm{D}^2-\alpha^2-\beta^2)^2-\mathrm{i}\alpha(U-2c+\tilde{c})(\mathrm{D}^2-\alpha^2-\beta^2)+\mathrm{i}\alpha U''\right\}\hat{w}_1^{(1)}$$
$$=-4\alpha^2\{\Phi_2^{(0)}(\mathrm{D}^2+\alpha^2+\beta^2)+\mathrm{D}\Phi_2^{(0)}\mathrm{D}\}\tilde{u}^{(0)}$$
$$-\{2\mathrm{i}\alpha\Phi_2^{(0)}(\mathrm{D}^2-\alpha^2-\beta^2)\mathrm{D}+3\mathrm{i}\alpha\mathrm{D}\Phi_2^{(0)}(D^2-\alpha^2-\beta^2)$$
$$+2\mathrm{i}\alpha D^2\Phi_2^{(0)}\mathrm{D}+\mathrm{i}\alpha\mathrm{D}^3\Phi_2^{(0)}\}\tilde{w}_1^{(0)}-\mathrm{i}\hat{\lambda}_1(\mathrm{D}^2-\alpha^2-\beta^2)w_1^{(0)}, \quad (3.8)$$

which contain the unknown constants λ_1 and $\hat{\lambda}_1$ in the forcing terms. The constant λ_1 in (3.7) can be obtained from the well-known solvability condition in the singular case of $c_i = 0$, where the inhomogeneous equation has the same differential operator as (3.6) and then the forcing terms must be orthogonal to the eigenfunction adjoint to $w_1^{(0)}$, while λ_1 in the other case of $c_i \neq 0$ is determined through the normalization imposed on $w_1^{(1)}$ in a form similar to (3.3). We also find that (3.8) belongs to the latter case, because the value of $2c-\tilde{\tilde{c}}$ cannot in general coincide with the eigenvalue $\hat{c}$, thus indicating no singularity of the equation. Knowing the numerical values of the Landau-type coefficients λ_1 and $\hat{\lambda}_1$, we can discuss various effects of weak two-dimensional waves on the development of three-dimensional disturbances on the basis of the amplitude equation (3.5), where λ_1 indicates the effects through two-dimensional distortion of the mean flow, and $\hat{\lambda}_1$ the effects through the second harmonic in two-dimensional waves. For the details of such a discussion, see Itoh (1980).

3.2. *Craik's resonant triads*

Following the basic idea of Raetz (1959), Craik (1971) considered three-dimensional disturbances consisting of two oblique waves which propagate at equal and opposite angles to the flow direction and whose streamwise wavenumber is just half that of the two-dimensional wave existing beforehand. He has shown that all three waves, when of the same phase velocity, in the downstream direction, construct a resonant triad, which leads to rapid growth of the oblique waves. To analyse such a situation, we may consider two-dimensional waves with the fundamental wavenumber 2α and ignore all components for odd k in (2.3), while only the components of the wavenumber α in the Fourier series (2.5) need be taken into account. Then an amplitude expansion of the disturbance gives rise to the series solution (3.4) and (3.5) with $v_1^{(1)} \equiv \lambda_1 \equiv 0$ and with A^2 replaced by A, which here denotes the complex amplitude of the two-dimensional wave Ψ_2. Coefficients in the series are determined from solution of (3.6) and (3.8). It should be noted that, since c in this formulation denotes the complex phase velocity of the two-dimensional wave with the wavenumber 2α, the inhomogeneous equation (3.8) can be of the singular type satisfying the condition $2c-\tilde{\tilde{c}} = \hat{c}$, under which $\hat{\lambda}_1$ must be determined from the solvability condition.

Putting

$$|A| = a, \quad |B| = b, \quad 2\arg(B)-\arg(A) = \theta, \quad (3.9)$$

and separating the modified amplitude equation (3.5) into real and imaginary parts, the equations governing the real amplitude b and the phase difference θ are

$$\frac{1}{b}\frac{\mathrm{d}b}{\mathrm{d}t} = \alpha\hat{c}_i-|\hat{\lambda}_1|\,a\sin(\theta-\theta_0)+O(a^2), \quad (3.10)$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -2\alpha(\hat{c}_r-c_r)-2|\hat{\lambda}_1|\,a\cos(\theta-\theta_0)+O(a^2), \quad (3.11)$$

where $\theta_0 = \arg(\hat{\lambda}_1)$. We now assume the amplitude a to be so small that the terms of $O(a^2)$ may be ignored, and consider Craik's resonant case of $\hat{c}_r = c_r$. Then the phase difference reaches a stable equilibrium at $\theta = \theta_0 - \frac{1}{2}\pi$, where the growth rate of b becomes maximum. Even in a non-resonant case with the restriction $\alpha|\hat{c}_r - c_r| < |\hat{\lambda}_1|\, a$, θ has a stable equilibrium value but the effect of a on the growth of b in this case is smaller than that in the resonant case. Furthermore, if $\alpha|\hat{c}_r - c_r| > |\hat{\lambda}_1|\, a$, then θ increases or decreases monotonically, so that the second term on the right-hand side of (3.10) changes its sign periodically, indicating that the effect of a will be averaged out to zero during a period θ. Consequently the most amplified three-dimensional wave is the one having the same phase velocity as the two-dimensional wave, and its spanwise wavenumber β is obtained by solving the relation

$$\hat{c}_r(\alpha, \beta, R) = c_r(\alpha, R), \tag{3.12}$$

provided that temporal variation of a is sufficiently moderate and that the Landau-type coefficient $\hat{\lambda}_1$ varies only slightly with β in an appropriate range.† It should be noted, however, that this kind of resonance can occur in a boundary layer, but not in plane Poiseuille flow, because its symmetric velocity profile gives the least stable eigenfunctions $\Phi_2^{(0)}$ and $w_1^{(0)}$ which are even functions of z, and then $\hat{\lambda}_1$ is found to be zero.

3.3. *Stability calculations of Herbert's model*

The above two methods are essentially based on weakly nonlinear theory and are concerned with a single component in the double Fourier series of three-dimensional disturbances. In contrast, Herbert (1984*a*) considered a two-dimensional unsteady flow consisting of laminar Poiseuille flow plus travelling waves with finite amplitude, which were numerically obtained by solving the full nonlinear equations with the use of iteration, and he investigated its linear stability to a group of three-dimensional waves with various x-Fourier components.

The most important point in Herbert's method is the assumption that two-dimensional waves are in equilibrium. Under this assumption, the Fourier coefficients in (2.3) are functions of z only, and with truncation of the series at an appropriate term $k = K$, we can determine Ψ_k $(k = 0, 1, \ldots, K)$ and the real phase velocity c from numerical solution of a finite set of ordinary differential equations. After applying similar truncation to (2.5), Herbert seeks a solution to the disturbance equations of the form

$$v_k(z, t) = \hat{v}_k(z)\, e^{st}, \tag{3.13}$$

where s is a complex constant independent of k. Substitution of this into (2.6)–(2.9) results in an eigenvalue problem to determine s, whose real part s_r denotes the temporal growth rate of the whole disturbance, while the imaginary part s_i indicates the difference in phase velocity from the two-dimensional wave.

If the two-dimensional waves are assumed to have the fundamental wavenumber 2α, Ψ_k for odd k being equal to zero, as was done in §3.2, it is then found that the disturbance equations for even k do not interact with those for odd k, each set of equations constructing an independent problem. The former set describes a three-

† Craik (1971) expected that $\hat{\lambda}_1$ would take a particularly large value when $\hat{c}_r = c_r$, because the Landau-type coefficient in that case was determined subject to the solvability condition, provided the two linear growth rates $\hat{c}_i$ and c_i were both small enough. Even in non-singular cases, however, $\hat{\lambda}_1$ can be determined from appropriate normalization conditions such as (3.3) and its value tends to the value at the singular point as $2c - \tilde{c}$ approaches $\hat{c}$ (see Itoh 1984). Thus there is no substance in the above argument.

dimensional disturbance with the same fundamental wavenumber 2α as two-dimensional waves, and therefore may be considered as an extension of the Stuart–Itoh formulation, while the latter corresponds to Craik's resonant theory, because its fundamental wavenumber is half that of the two-dimensional wave. Herbert (1984*a*, 1985) numerically solved the two kinds of eigenvalue problem for plane Poiseuille flow and the Blasius boundary layer, and demonstrated the so-called 'threshold amplitude' of two-dimensional waves, above which the disturbances grow in time.

In the theoretical procedure mentioned above, an objection may be addressed to the assumption that all the Fourier components of the disturbance have the same temporal variation, as indicated in (3.13). This assumption is rational only when two-dimensional waves are rigorously in equilibrium, and also when the disturbance is so small that the governing equations may be linearized. Otherwise, each Fourier component of the disturbance will have different dependence on time owing to different interactions with various components, and the above formulation will break down. Since the theory, at least in the present form, cannot be extended to a slightly non-equilibrium state of two-dimensional waves, we should consider that analyses have been made of a mathematical model rather than an approximation to a real flow field. Nevertheless, it is certain that Herbert's pioneering work has made important progress in understanding the physical phenomena concerned here.

4. Motivation and basic idea

We have seen that existing theories give a variety of solutions to the disturbance-equation system (2.6)–(2.9) subject to different conditions of the two-dimensional wave and three-dimensional disturbances. The above references seem to provide sufficient theoretical bases for the new analysis given later, although there are other attractive or advanced studies relating to the problems of three-dimensionality, for instance those by Nayfeh (1981), Craik (1982), Dhanak (1983) and Benney (1984). Since the two-dimensional wave is considered to be of finite magnitude, all the Fourier components should be taken into account, at least in principle. Concerning the disturbance, however, there is a choice of some particular components dominating the whole. It is of course possible to take all the Fourier components, as was done by Herbert (1984*a*). This approach resembles numerical simulations and may be useful in describing the quantitative and fully detailed behaviour of the disturbance, but seems not to be convenient for revealing the most essential feature of the instability mechanism, which will be buried in very complicated interactions among all the Fourier components of the basic flow and the disturbance. In this paper, therefore, attention is directed to a single Fourier component of the disturbance, and its interaction with related components of the basic flow is investigated.

For the present purpose of theoretical understanding of important factors affecting the main development of instability waves, it will be helpful to clarify the most important outcome of the weakly nonlinear theories based on a series expansion of the disturbance in powers of the amplitude of a two-dimensional wave, although this approach has the great disadvantage of a small radius of convergence of the series solution. In the method of amplitude expansion, the lowest-order approximation to the solution is usually given by an eigensolution of the Orr–Sommerfeld equation, and higher-order corrections are governed by inhomogeneous equations. The forcing terms, originating in quadratic terms of the Navier–Stokes equations, are produced through interactions among various Fourier components in two-dimensional waves and disturbances. If a complex frequency of the forcing term coincides with an eigenvalue of the corresponding homogeneous equation, the problem becomes singular and has a solution only if the forcing term satisfies a solvability condition, as mentioned in the previous section. Thus the solvability condition may be considered as a mathematical expression of the physical situation of resonance between a forcing term and an eigensolution. The analyses of Stuart (1962) and Itoh (1980) deal with the case where the forcing terms induced by couplings between the zeroth Fourier component of two-dimensional waves and the fundamental of three-dimensional disturbances, and between the second harmonic of the waves and the fundamental of the disturbances, are in resonance or near resonance with the corresponding eigensolution. Also, Craik's (1971) work is concerned with resonance between the forcing term due to interaction of the two-dimensional wave with a three-dimensional wave of half the streamwise wavenumber and the three-dimensional eigensolution.

Besides these two examples, there will be some resonant or nearly resonant states in the sense described above. For instance, the equations for mean-flow distortion (zeroth Fourier component) may admit a resonant solution, because their eigensolutions have very small damping factors, that is, of the same order as growth rates of Tollmien–Schlichting waves, as frequently pointed out by Davey & Nguyen (1971), Itoh (1974*b*), Herbert (1980), Mizushima & Gotoh (1985) and others in the framework of two-dimensional problems. In those cases, equations of the form (2.6) with $k = 0$ and $\beta = 0$ play fundamental roles.

On the other hand, a resonance inherent in three-dimensional problems may possibly occur in relation to the additional equation (2.7). In fact, Benney & Gustavsson (1981) investigated the equation for $k = 1$ in detail and accentuated the importance of this kind of resonance. In the present problem, however, the two equations (2.6) and (2.7) for $k = 1$ both have forcing terms that are induced by interaction of the wavenumber vectors $(-\alpha, \beta)$ and $(2\alpha, 0)$. Then the forced solution of the first equation will behave according to the theory of Stuart or Craik mentioned above. In such a situation, we hesitate to claim that the Benney–Gustavsson resonance associated with the second equation is of particular importance.

At this stage, it may be natural to notice that a new type of mechanism similar to the Benney–Gustavsson resonance may act on the mean-flow field in the process of three-dimensionalization of instability waves. If only the mean-flow term of the disturbance is retained, we have the homogeneous form of (2.6) and an inhomogeneous form of (2.7) for $k = 0$. Thus we can expect that the second equation leads to a simple but very important resonance.

The next section is devoted to pursuing this possibility.

5. Analysis of the mean-flow field

Following the above deduction, we assume here that only the mean-flow term with $k = 0$ in the Fourier series (2.5) of the spanwise-periodic three-dimensional disturbance is dominant, the other components being negligibly small. This assumption allows us to put $k = l = 0$ in (2.6)–(2.9) to yield the simplified equation system

$$\left\{\frac{1}{R}(\mathrm{D}^2-\beta^2)-\frac{\partial}{\partial t}\right\}(\mathrm{D}^2-\beta^2)\,w_0 = 0, \tag{5.1}$$

Eigenfunction	Characteristic equation	Maximum eigenvalue
$\psi(z)$		
Even	$p\tan p = -\beta\tanh\beta$	$-\frac{1}{R}(\beta^2+\frac{1}{4}\pi^2) > \rho_0 > -\frac{1}{R}(\beta^2+\pi^2)$
Odd	$\frac{\tan p}{p} = \frac{\tanh\beta}{\beta}$	$-\frac{1}{R}(\beta^2+\pi^2) > \rho_0 > -\frac{1}{R}(\beta^2+\frac{9}{4}\pi^2)$
$\phi(z)$		
Even	$\cos q = 0$	$\sigma_0 = -\frac{1}{R}(\beta^2+\frac{1}{4}\pi^2)$
Odd	$\sin q = 0$	$\sigma_0 = -\frac{1}{R}(\beta^2+\pi^2)$

TABLE 1. Eigenvalues of the mean-flow equations ($p \equiv -\rho R-\beta^2 > 0,\quad q \equiv -\sigma R-\beta^2 > 0$)

$$\left\{\frac{1}{R}(\mathrm{D}^2-\beta^2)-\frac{\partial}{\partial t}\right\}u_0 = U'w_0+\Psi_0''w_0, \tag{5.2}$$

$$\beta v_0+\mathrm{D}w_0 = 0, \tag{5.3}$$

$$u_0 = w_0 = \mathrm{D}w_0 = 0 \quad \text{at} \quad z = \pm 1, \tag{5.4}$$

where $U(z)$ denotes the velocity of steady laminar flow and is given by $U = 1-z^2$ in the problem of plane Poiseuille flow, while $\Psi_0(z,t)$ is the stream function of the mean-flow distortion induced by nonlinear interaction of two-dimensional waves. Since the most amplified fundamental Ψ_1 in this flow is an even function of z, the mean-flow term Ψ_0 may be assumed to be an odd function of z (see Stuart 1960).

It is first necessary to obtain eigensolutions of (5.1) and (5.2) neglecting the forcing terms. From the consistency with the continuity equation (5.3), we put the solutions in the form

$$w_0(z,t) = -b_0\beta\psi(z)\,\mathrm{e}^{\rho t}, \quad u_0(z,t) = a_0\phi(z)\,\mathrm{e}^{\sigma t}, \tag{5.5}$$

where a_0 and b_0 are arbitrary constants. The procedure for obtaining the eigenvalues ρ and σ is obvious and so the main results are presented in table 1. All eigenvalues of the mean-flow equations are real and negative, and the corresponding eigenfunctions are classified into odd and even functions. In the stability analysis concerned here, only the maximum eigenvalues ρ_0 and σ_0 in the two classes need be taken into account.

Next, we direct our attention to the inhomogeneous equation (5.2) and reveal the possibility of resonance there, as mentioned earlier. Although we have two forcing terms on the right-hand side, the first term seems not to be important for the present purpose, because its time dependence comes from the exponential term $\mathrm{e}^{\rho_0 t}$ of w_0 and so is always different from that of the eigensolution of u_0, as seen in table 1. On the other hand, the second forcing term has the possibility of resonance with the eigensolution, because Ψ_0 as well as w_0 there will vary with time and, at least under the weakly nonlinear approximation, the total dependence is written in an exponential function. Thus analysis hereafter will be made on the solution corresponding to the second forcing term in (5.2), neglecting the first one.

On the assumption that the two-dimensional distortion of the mean flow is written in the form

$$\Psi_0 = |A|^2\,\Phi_0(z)\,\mathrm{e}^{(s-\rho_0)t}, \tag{5.6}$$

we have the following equation to solve:

$$\left\{\frac{1}{R}(\mathrm{D}^2-\beta^2)-\frac{\partial}{\partial t}\right\}u_0 = -b_0|A|^2\beta\Phi_0''(z)\,\psi_0(z)\,\mathrm{e}^{st}, \tag{5.7}$$

where $|A|$ denotes the amplitude of Tollmien–Schlichting waves and the real constant s represents the total growth rate of the forcing term. The general solution will be given by the sum of a particular solution and an infinite sequence of eigensolutions, but here we consider a particular situation where the constant s has a slightly positive or negative value close to the maximum eigenvalue σ_0. As mentioned earlier, the growth rate αc_i of the Tollmien–Schlichting wave is generally small in magnitude, and of the same order as ρ_0 and σ_0. If we apply the weakly nonlinear theory to the solution of (2.4), for simplicity, the growth rate of Ψ_0 is found to be given by twice that of Ψ_1, indicating that the situation where $s = 2\alpha c_i+\rho_0$ is very close to σ_0 actually happens for a small value of αc_i, that is, for a slowly growing or decaying Tollmien–Schlichting wave. Then we may ignore all the decaying eigensolutions except the one belonging to σ_0, and write an approximate solution to (5.7) in the form

$$u_0 = -b_0|A|^2\beta\left\{\lambda_0\,\phi_0(z)\frac{\mathrm{e}^{st}-\mathrm{e}^{\sigma_0 t}}{s-\sigma_0}+f(z)\,\mathrm{e}^{st}\right\}, \tag{5.8}$$

where the function $f(z)$ is required to be orthogonal to the self-adjoint eigenfunction $\phi_0(z)$, because it represents the remainder after subtracting the ϕ_0 component from the eigenfunction expansion of the particular solution. Substitution of (5.8) into (5.7) yields

$$\left\{\frac{1}{R}(\mathrm{D}^2-\beta^2)-s\right\}f = \Phi_0''\psi_0+\lambda_0\phi_0, \tag{5.9}$$

which has a solution if the Landau-type constant λ_0 is given by

$$\lambda_0 = -\frac{\int_0^1 \phi_0(z)\,\Phi_0''(z)\,\psi_0(z)\,\mathrm{d}z}{\int_0^1 \{\phi_0(z)\}^2\,\mathrm{d}z}. \tag{5.10}$$

It is, of course, allowable to add the ϕ_0 eigensolution with an arbitrary coefficient to the right-hand side of (5.8), but the additional term decays exponentially with time without any significant contribution, and so may be eliminated by imposing an artificial condition that the ϕ_0 component in the whole solution is negligibly small at the initial time $t = 0$. In fact, the solution (5.8) contains the most important part describing the main development of the velocity u_0, because the ϕ_0 term retained there is found to become dominant in a finite range of time, provided s is sufficiently close to σ_0. To see this, we introduce an amplitude function defined by

$$a(t) = -b_0|A|^2\beta\lambda_0\frac{\mathrm{e}^{st}-\mathrm{e}^{\sigma_0 t}}{s-\sigma_0}, \tag{5.11}$$

and investigate its variation with time.

Let us consider first the limiting case of $s\to\sigma_0$, which indicates the resonance in our sense. Then (5.11) reduces to

$$a(t) = -b_0|A|^2\beta\lambda_0 t\,\mathrm{e}^{\sigma_0 t}, \tag{5.12}$$

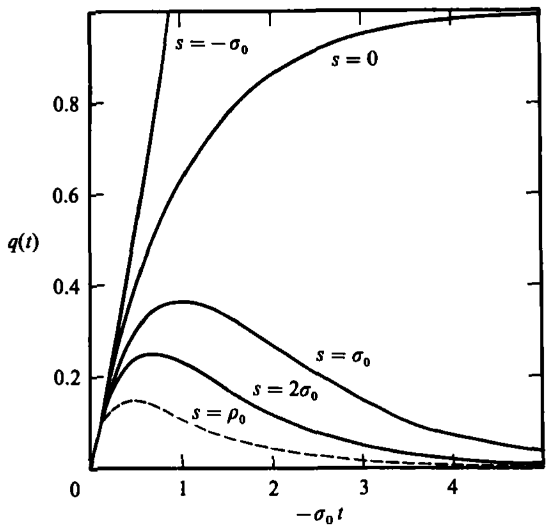

FIGURE 1. Behaviour of the amplitude function for various values of s: $q(t) \equiv -\sigma_0 a(t)/(b_0|A|^2\beta\lambda_0) = \sigma_0(e^{st}-e^{\sigma_0 t})/(s-\sigma_0)$. The broken line denotes the case of $\rho_0 = -0.4264\times 10^{-2}$, $\sigma_0 = -0.1005\times 10^{-2}$, $R = 5000$ and $\beta = 1.60$.

and shows that the amplitude begins with $a = 0$ at $t = 0$, attains the maximum value

$$a_{\max} = \frac{b_0|A|^2\beta\lambda_0}{\sigma_0 \mathrm{e}} \tag{5.13}$$

at the time $t = t_{\max} \equiv -1/\sigma_0$, and finally decays to zero as $t\to\infty$. Since the maximum amplitude given above is proportional to the reciprocal of σ_0, the most important contribution to the disturbance development seems to be made by the eigensolution with $|\sigma_0|$ minimum. Thus we choose the case of ϕ_0 even in table 1, and put

$$\sigma_0 = -\frac{1}{R}(\beta^2+\tfrac{1}{4}\pi^2), \quad \phi_0(z) = \cos\tfrac{1}{2}\pi z. \tag{5.14}$$

Then ψ_0 must be the odd eigenfunction belonging to the smaller eigenvalue of ρ_0 given in table 1, because an even forcing term is needed to yield an even solution of (5.7). The velocity u_0 due to the even eigenfunction above has the same symmetry about the channel centre as the two-dimensional mean flow, and therefore may be considered to represent a small spanwise variation in magnitude of the original flow, being consistent with various experimental observations.

Next, we examine the nearly resonant cases, where s is not strictly equal to σ_0 but $|s-\sigma_0|$ is sufficiently small. If s is negative, the amplitude function varies with time in a way similar to that of the resonant case. If s is zero or has a slightly positive value, however, the amplitude does not decay to zero for large time but increases monotonically with time. Figure 1 shows variations of $a(t)$ with time for the finally stable ($s < 0$), neutral ($s = 0$) and unstable ($s > 0$) cases. Thus we can draw the important conclusion that if s is positive, that is, if Ψ_0 has a growth rate larger than $-\rho_0$, an initial three-dimensionality, measured by b_0 in the present formulation, induces an exponential growth of the spanwise-periodic streamwise flow u_0. However, it should be noticed, from considering the more practical problem of spatial dependence, that what is really important for our present purpose is not the final state of infinite time but the maximum value that the amplitude can attain during

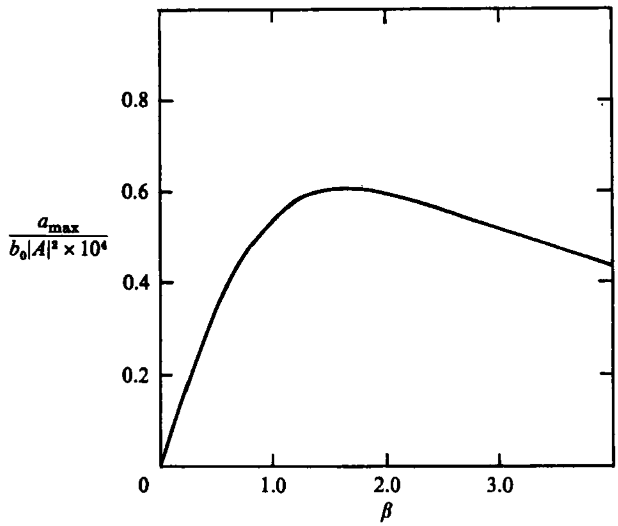

FIGURE 2. Variation of a_{max} with β for the case of $R = 5000$ and $\alpha = 1.2$.

a finite time. Even in the case where s is slightly negative (including the resonant case), the maximum of $a(t)/(b_0|A|^2)$ is numerically very large because of the resonant or nearly resonant state. When $b_0|A|^2$ is very large, therefore, $a(t)$ can grow near the maximum point to a magnitude sufficiently large for its nonlinear effects to keep the flow field three-dimensional. Thus we may use a_{max} given by (5.13) as a measure of growth of three-dimensionality, without specification of a value of s. It may be noted here that the asymptotic amplitude for large time of the neutrally stable solution with $s = 0$ in figure 1 differs from a_{max} only by the constant factor $e = 2.718 \ldots$.

For further discussion on the amplitude function, we need to know values of the Landau-type constant λ_0, which can be obtained by solving (5.9) with the author's (Itoh 1974*a*) numerical method of solution. The two-dimensional mean flow of the form $\Psi_0 = |A|^2 \Phi_0(z)$ is obtained from the weakly nonlinear method. In (2.4), we put $k = \partial/\partial t = 0$, substitute $A\Phi_1^{(0)}(z)$ and its complex conjugate into Ψ_1 and Ψ_{-1} and neglect other Fourier components on the right-hand side. For definition of the amplitude A and of b_0, the eigenfunctions of the Orr–Sommerfeld equation and (5.1) are normalized as $\Phi_1^{(0)}(0) = 1$ and $\psi_0(0) = 1$. Numerical results thus obtained indicate that λ_0 is not sensitive to variation of the spanwise wavenumber β, and therefore the dependence of a_{max} on β is mainly due to the β/σ_0 term in (5.13). Actually the curve of $a_{max}/(b_0|A|^2)$ plotted against β for a fixed value of the Reynolds number has a maximum near $\beta = \frac{1}{2}\pi$, as shown in figure 2. This peak, which represents an amplification factor of the most unstable three-dimensional disturbance, is not very sharp but seems at least to suggest the existence of a preferred range of spanwise wavenumber in the process of three-dimensionalization of Tollmien–Schlichting waves. The main features of this figure seem to be unchanged for quite general cases with different values of the Reynolds number R and the two-dimensional wavenumber α, because the fact that λ_0 varies only slightly with β will normally be true in those cases.

It should be emphasized again that we can understand, from values of a/b_0, how much the three-dimensional components originally included in the mean flow are amplified through the above mechanism of resonance under the existence of two-dimensional waves. Figure 2 indicates that the amplification factor remains of order unity if the amplitude of the Tollmien–Schlichting waves is less than 2 %, but that the factor can become one order of magnitude greater than unity for an amplitude above 4 %.

Finally a remark should be made upon the first forcing term in (5.2). Although neglected in the preceding analysis, this term is much larger in magnitude than the second forcing term, and may also be considered close to resonance, because $|\rho_0|$ and $|\sigma_0|$ are both numerically small. Indeed we have a nearly resonant solution, whose Landau-type constant $\bar{\lambda}_0$ is obtained by replacing Φ_0'' with U' in (5.10) and found to be much smaller in magnitude than λ_0; for instance, $\bar{\lambda} = -0.3059$ and $\lambda_0 = -10.38$ for the case of $R = 5000$, $\alpha = 1.20$ and $\beta = 1.60$, indicating that the two forced solutions may be considered roughly of a comparable order when the amplitude of Tollmien–Schlichting waves is larger than 4 % or so. Since the first forcing term has a negative growth rate a little away from the resonance, the corresponding solution attains its maximum in a rather short time and then decays slowly (see figure 1). Therefore the total behaviour of u_0 will be dominated in the earlier stage by the first solution, but later by the solution corresponding to the second forcing term if the growth rate s is positive or near zero. It may also be noted that the first solution does not involve the Tollmien–Schlichting waves and so describes the behaviour of a small three-dimensionality introduced into a steady laminar flow. Its slow damping for large time may explain the experimental difficulty of complete elimination of minute spanwise variation in the basic flow encountered by Nishioka & Asai (1985).

6. Concluding remarks

The analysis given in the previous section has predicted a strong possibility that, when a certain magnitude of Tollmien–Schlichting waves exists, three-dimensional distortion with a spanwise periodicity is produced in the mean-flow field. It is naturally desirable to directly compare this prediction with experiments, but no appropriate experimental data are available, because detailed measurement of the mean-flow field is very difficult when quite large travelling waves exist. Fortunately, however, the mean-flow distortion is considered to grow rather rapidly, becoming so large that its interaction with the existing two-dimensional waves will become substantial. This coupling yields three-dimensional travelling waves with the same streamwise wavenumber as that of Tollmien–Schlichting waves, and with the same spanwise wavenumber as that of the mean-flow distortion, resulting in the peak–valley splitting observed by Klebanoff *et al.* (1962) in the famous experiments on the flat-plate boundary layer. Although the mean-flow distortion varies with time in a manner slightly different from an exponential function, as seen previously, we simply apply the equation system (2.6)–(2.9) to calculate the velocity distribution of the resulting three-dimensional waves. Putting $k = 1, \partial/\partial t \equiv 0$, and substituting $\Phi_1^{(0)}(z)$ and $\phi_0(z)$ into Ψ_1 and u_0 respectively, on the right-hand side, with neglect of other Fourier components, leads to a set of simplified equations, which can be solved numerically. The velocity and phase distributions of u_1 thus obtained are compared with those of the two-dimensional solution $U_1 = \mathrm{D}\Phi_1^{(0)}(z)$ in figure 3. The main differences are in the location of the maximum amplitude and in the phase shift in the neighbourhood of the wall. These features of the distributions of the velocity and phase agree very well with the experimental results of Nishioka & Asai (1985, figures 6 and 7).

As mentioned earlier, there are two types of three-dimensional development of Tollmien–Schlichting waves, one being the formation of Klebanoff's peak–valley

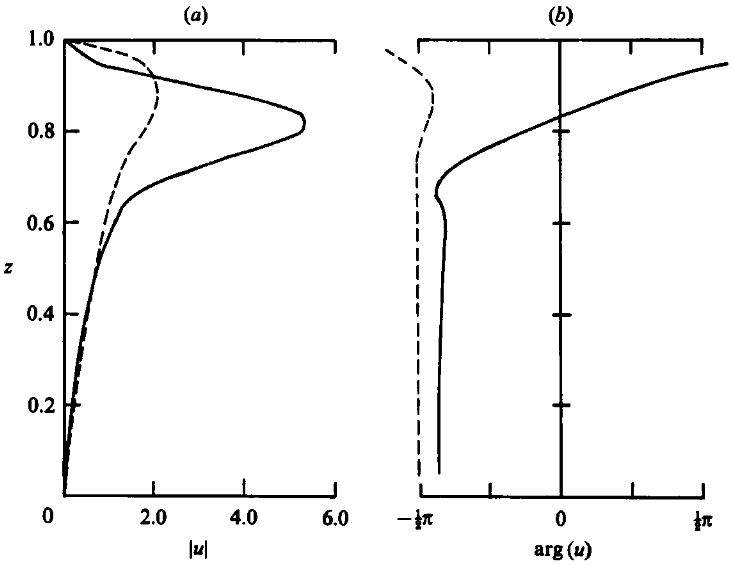

FIGURE 3. (*a*) Velocity and (*b*) phase distributions of the three-dimensional travelling wave (——) and the Tollmien–Schlichting wave (– – –). $R = 5000$, $\alpha = 1.2$ and $\beta = 1.6$.

splitting, the other of the staggered type theoretically predicted by Craik (1971). Stability calculations made by Herbert (1984*a*, 1985) have clearly indicated that the unsteady flow consisting of a laminar flow plus Tollmien–Schlichting waves with finite amplitude is unstable to both types of three-dimensional disturbance. In contrast with the staggered type caused by Craik's resonant triads, however, the mechanics leading to the peak–valley splitting has awaited satisfactory explanation for a long time, although some attempts were made to this end by Stuart (1962) and Itoh (1980). It may therefore be expected that an advance in understanding of the phenomena will be stimulated by the mechanics proposed in the present paper, with accentuation of the importance of the mean-flow field.

In this paper, the temporally dependent approach has been used only for mathematical simplicity. However, essential features in the analysis would be similar if the roles of the time t and the streamwise distance x were exchanged to give preference to the spatially dependent approach, which might be convenient for comparison with experimental observations. It is also for mathematical simplicity that plane Poiseuille flow was chosen as the basic laminar flow. Some modifications will be necessary for application of the theory to the problem of boundary-layer flows because of the weak but important effects of non-parallelism on the mean-flow field (Itoh 1974*c*, 1984).

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